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The Klein–Gordon Equation as a Limiting Case of a Suitably Hyperbolized Schrödinger Equation†

F. BORGHESE, P. DENTI
Istituto di Fisica dell'Universitá di Messina, 98100 Messina, Italy

and T. RUGGERI
Istituto di Matematica dell'Universitá di Messina, 98100 Messina, Italy

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Abstract

We obtain a hyperbolic equation whose discontinuity waves are all exceptional and propagate with velocity λ. When $\lambda \to \infty$ or $\lambda = c$, this equation becomes identical to the Schrödinger equation and to the Klein–Gordon equation respectively. We also show that λ is related to the dispersion relation $E(p)$.

1. *Introduction*

It is well known that the Klein–Gordon equation is obtained from the relativistic Hamiltonian through the Schrödinger representation. Indeed, the above equation is hyperbolic whereas the Schrödinger equation is parabolic so that it is impossible for the latter equation to transform continuously into the former. A similar problem arises in generalizing to the relativistic case the equation of heat propagation (Boillat, 1970; Boillat & Ruggeri, 1971). Bearing in mind the solution of this latter problem, we build up a hyperbolic equation by adding to the Schrödinger equation a term of the form

$$\alpha_{11}\frac{\partial}{\partial t}F\left(\frac{\partial\psi}{\partial t}\right) + \alpha_{01}F\left(\frac{\partial\psi}{\partial t}\right) + \alpha_{00}F(\psi)$$

the $\alpha_{i\kappa}$'s being suitably chosen constant coefficients. We require the solutions of the hyperbolized equation to be always everywhere continuous together with their first derivatives, according to the usual quantum conditions. This decrees that the discontinuity waves associated with the

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above equation never evolve into shocks, i.e. they are all exceptional in the sense of Lax (1954) and Boillat (1965).

The function F can thus be determined and one obtains a linear equation according to the superposition principle. The coefficients of the resulting equation depend on the propagation velocity of the discontinuity waves, λ, and when $\lambda = c$, the velocity of light, the equation becomes covariant and coincides with the Klein–Gordon equation; when $\lambda \to \infty$ the equation has as a limit the Schrödinger equation. A continuous connection between these two equations has thus been obtained.

2. *Hyperbolization of the Schrödinger Equation*

Let us consider the differential equation

$$-\frac{\hbar^2}{2m}\nabla^2\psi = i\hbar\frac{\partial\psi}{\partial t} + \frac{\partial}{\partial t}F\left(\frac{\partial\psi}{\partial t}\right) + \frac{2imc^2}{\hbar}F\left(\frac{\partial\psi}{\partial t}\right) + \frac{m^2c^4}{\hbar^2}F(\psi) \tag{2.1}$$

where the function $F(x)$ is chosen so that $F(0) \equiv 0$ and equation (2.1) be a hyperbolization of the Schrödinger equation ($F \equiv 0$).

We introduce the quantities

$$\mathbf{v} = \nabla\psi, \qquad \phi = \frac{\partial\psi}{\partial t} \tag{2.2}$$

in terms of which we obtain

$$\begin{cases} -\dfrac{\hbar^2}{2m}\left(\nabla.\mathbf{v} + \dfrac{2m}{\hbar^2}F'(\phi)\dfrac{\partial\phi}{\partial t}\right) = i\hbar\phi + \dfrac{2imc^2}{\hbar}F(\phi) + \dfrac{m^2c^4}{\hbar^2}F(\psi) \\ \dfrac{\partial\mathbf{v}}{\partial t} = \nabla\phi \end{cases} \tag{2.3}$$

We assume that ψ, ϕ and $\mathbf{v}$ are everywhere continuous together with $F(x)$ and its derivatives, whereas on the surface

$$\sigma(\mathbf{r}, t) = 0$$

the derivatives of $\mathbf{v}$ and ϕ are not. Let us put (Donato & Ruggeri, 1972)

$$\mathbf{n} = \frac{\nabla\sigma}{|\nabla\sigma|}, \qquad -\lambda = \frac{\partial\sigma}{\partial t}\frac{1}{|\nabla\sigma|}, \qquad \delta = \left(\frac{\partial}{\partial\sigma}\right)_+ - \left(\frac{\partial}{\partial\sigma}\right)_-$$

where $\mathbf{n}$ is the unit normal to σ, λ is the normal propagation velocity of σ and δ indicates the jump of derivatives across σ.

The equation for the discontinuities can be written through the substitution

$$\frac{\partial}{\partial t} \to -\lambda\delta, \qquad \nabla \to \mathbf{n}\delta$$

which yields

$$\begin{cases} \mathbf{n}\,.\,\delta\mathbf{v} - \dfrac{2m}{\hbar^2} F'(\phi)\,\lambda\delta\phi = 0 \\ -\lambda\delta\mathbf{v} = \mathbf{n}\delta\phi \end{cases} \tag{2.4}$$

Setting aside the case $\lambda = 0$ ($\delta\phi = 0$), we obtain from equation (2.4)

$$\left(1 + \frac{2m}{\hbar^2} F'(\phi)\,\lambda^2\right)\delta\phi = 0$$

and thus

$$\lambda^2 = -\frac{\hbar^2}{2mF'(\phi)}$$

Since equation (2.1) is assumed to be hyperbolic, λ should be real and thus $F'(\phi) < 0$. Furthermore we recall that a wave is said to be exceptional (Boillat, 1965) when $\delta\lambda \equiv 0$, i.e. in our case $F''(\phi) = 0$. Therefore $F'(\phi)$ is constant together with λ^2, and, being $F(0) = 0$,

$$F(x) = -\frac{\hbar^2}{2m\lambda^2} x \tag{2.5}$$

By substituting (2.5) into (2.1) we get the final equation

$$-\frac{\hbar^2}{2m}\left(\nabla^2 - \frac{1}{\lambda^2}\frac{\partial^2}{\partial t^2}\right)\psi = \frac{i\hbar}{\lambda^2}(\lambda^2 - c^2)\frac{\partial\psi}{\partial t} - \frac{mc^4}{2\lambda^2}\psi \tag{2.6}$$

When $\lambda \to \infty$ equation (2.6) reduces to the Schrödinger equation, whereas when $\lambda = c$ it becomes covariant and coincides with the Klein–Gordon equation.

3. *Relation between λ and $E(p)$*

The role of λ can be better understood by searching for solutions of (2.6) in the form of plane waves

$$\psi(x, t) = A \exp\,[i(px - Et)/\hbar]$$

One obtains at once

$$p(2mv - p)\,\lambda^2 = m^2 c^4 + 2mc^2 pv - p^2 v^2 \tag{3.1}$$

where $v = E/p$ is the vibration velocity. From (3.1) one has

$$\lambda^2 = \frac{E^2 - 2mc^2 E - m^2 c^4}{p^2 - 2mE} \tag{3.2}$$

which relates λ to the energy and momentum of the particle.

This relation, of quantum origin, provides the correct dispersion relation $E(p)$ in both the non-relativistic ($\lambda \to \infty$) and in the relativistic limit ($\lambda = c$).

Conversely, if one assumes a given dispersion relation, obtains from (3.2) a value of λ which, introduced into (2.6), yields a solution which may be correct within the correctness of the dispersion relation one assumes.

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